

# A criterion for finite rank $\lambda$ -Toeplitz operators

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**AMS classification:** 47A05, 47B38

## Abstract

Let  $\lambda$  be a complex number in the closed unit disc  $\overline{\mathbb{D}}$ , and  $\mathcal{H}$  be a separable Hilbert space with the orthonormal basis, say,  $\mathcal{E} = \{e_n : n = 0, 1, 2, \dots\}$ . A bounded operator  $T$  on  $\mathcal{H}$  is called a  $\lambda$ -*Toeplitz operator* if  $\langle Te_{m+1}, e_{n+1} \rangle = \lambda \langle Te_m, e_n \rangle$  (where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{H}$ ). The subject arises naturally from a special case of the operator equation

$$S^*AS = \lambda A + B, \text{ where } S \text{ is a shift on } \mathcal{H},$$

which plays an essential role in finding bounded matrix  $(a_{ij})$  on  $l^2(\mathbb{Z})$  that solves the system of equations

$$\begin{cases} a_{2i,2j} &= p_{ij} + aa_{ij} \\ a_{2i,2j-1} &= q_{ij} + ba_{ij} \\ a_{2i-1,2j} &= v_{ij} + ca_{ij} \\ a_{2i-1,2j-1} &= w_{ij} + da_{ij} \end{cases}$$

for all  $i, j \in \mathbb{Z}$ , where  $(p_{ij})$ ,  $(q_{ij})$ ,  $(v_{ij})$ ,  $(w_{ij})$  are bounded matrices on  $l^2(\mathbb{Z})$  and  $a, b, c, d \in \mathbb{C}$ . It is also clear that the well-known Toeplitz operators are precisely the solutions of  $S^*AS = A$ , when  $S$  is the unilateral shift. In this paper we verify some basic issues, such as boundedness and compactness, for  $\lambda$ -Toeplitz operators and, our main result is to give necessary and sufficient conditions for finite rank  $\lambda$ -Toeplitz operators.

# 1 Introduction

Let  $\mathcal{H}$  be a separable Hilbert space with an orthonormal basis, say,  $\mathcal{E} = \{e_n : n = 0, 1, 2, \dots\}$ . Given  $\lambda \in \mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ , a bounded operator  $T$  is called a  $\lambda$ -*Toeplitz operator* if  $\langle Te_{m+1}, e_{n+1} \rangle = \lambda \langle Te_m, e_n \rangle$  (where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathcal{H}$ ). In terms of the basis  $\mathcal{E}$ , it is easy to see that the matrix representation of  $T$  is given by

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & a_{-4} & \cdots \\ a_1 & \lambda a_0 & \lambda a_{-1} & \lambda a_{-2} & \lambda a_{-3} & \ddots \\ a_2 & \lambda a_1 & \lambda^2 a_0 & \lambda^2 a_{-1} & \lambda^2 a_{-2} & \ddots \\ a_3 & \lambda a_2 & \lambda^2 a_1 & \lambda^3 a_0 & \lambda^3 a_{-1} & \ddots \\ a_4 & \lambda a_3 & \lambda^2 a_2 & \lambda^3 a_1 & \lambda^4 a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

for some double sequence  $\{a_n : n \in \mathbb{Z}\}$ , and the boundedness of  $T$  clearly implies that  $\sum |a_n|^2 < \infty$ . Therefore, it is natural to introduce the notation

$$T = T_{\lambda, \varphi},$$

where  $\varphi \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$  belongs to  $L^2 = L^2(\mathbb{T})$ , the Hilbert space of square integrable functions on the unit circle  $\mathbb{T}$ , with inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \bar{g} d\theta,$$

and consider  $T_{\lambda, \varphi}$  as an operator acting on  $H^2 \subseteq L^2$ , the *Hardy space*

$$\left\{ f \in L^2 : \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0, \ n < 0 \right\}$$

with the identification  $\mathcal{H} = H^2$  and  $e_n$  identified with the function  $e^{in\theta}$ ,  $n \geq 0$ . It is well-known that if  $f \sim \sum_0^{\infty} a_n e^{in\theta} \in H^2$ , then the analytic function

$$\hat{f}(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - ze^{-i\theta}} d\theta, \ |z| < 1$$

equals the analytic function defined by the series  $\sum_0^{\infty} a_n z^n$ ,  $|z| < 1$ , and, by a theorem of Fatou,  $\hat{f}_r(e^{i\theta}) = \hat{f}(re^{i\theta}) \rightarrow f(e^{i\theta})$  a.e.  $\theta$  and in  $L^2$  (and for this

reason  $f$  is sometimes called the *boundary value* of  $\hat{f}$ ). Hence  $H^2$  is often identified with the Hilbert space of analytic functions  $\{\hat{f} : f \in H^2\}$ , with inner product

$$\langle \hat{f}, \hat{g} \rangle = \frac{1}{2\pi} \int_0^{2\pi} f \bar{g} d\theta,$$

and we shall not make any distinction between  $\hat{f}$  and  $f$  throughout the rest of this paper. Also note that when  $\lambda = 1$  and  $\varphi \in L^\infty = L^\infty(\mathbb{T})$ , the matrix of  $T_{1,\varphi}$  is the matrix of the bounded Toeplitz operator  $T_\varphi f = P(\varphi f)$ ,  $f \in H^2$ , where  $P$  is the projection from  $L^2$  on to  $H^2$ . Here we refer the reader to [5] and [9], both of which are excellent sources of information on the theory of Hardy spaces and Toeplitz operators on  $H^2$ .

The subject “ $\lambda$ -Toeplitz operator” arises in a natural way from the study of the bounded matrix  $(a_{ij})$  on  $l^2(\mathbb{Z})$  that solves the system of equations

$$\begin{cases} a_{2i,2j} &= p_{ij} + aa_{ij} \\ a_{2i,2j-1} &= q_{ij} + ba_{ij} \\ a_{2i-1,2j} &= v_{ij} + ca_{ij} \\ a_{2i-1,2j-1} &= w_{ij} + da_{ij} \end{cases} \quad (\star)$$

for all  $i, j \in \mathbb{Z}$ , where  $(p_{ij})$ ,  $(q_{ij})$ ,  $(v_{ij})$ ,  $(w_{ij})$  are bounded matrices on  $l^2(\mathbb{Z})$  and  $a, b, c, d \in \mathbb{C}$  (See [8]). One of the major steps for analyzing the solutions of  $(\star)$  is to determine bounded operator  $A$  on  $\mathcal{H}$  satisfying the operator equation below:

$$S^*AS = \lambda A + B,$$

where  $S$  is a shift on  $\mathcal{H}$ ,  $B$  is fixed, and  $|\lambda| \leq 1$ . Notice that if we consider the map on  $\mathcal{B}(\mathcal{H})$ :

$$\phi(A) = S^*AS, \quad A \in \mathcal{B}(\mathcal{H}),$$

with  $S$  being the unilateral shift, i.e.,  $Se_n = e_{n+1}$ ,  $n = 0, 1, 2, \dots$ , then it is not difficult to see, by definition, that the  $\lambda$ -Toeplitz operators are the “eigenvectors” for  $\phi$  associated with  $\lambda$ . For instance, the Toeplitz operators are just the “eigenvectors” for  $\phi$  associated with the eigenvalue 1, and for every  $\lambda$  with  $|\lambda| = 1$ , the “eigenvectors” of  $\phi$  for the eigenvalue  $\lambda$  are the *Toeplitz-composition operators*, i.e., elements in the so-called *Toeplitz-composition  $C^*$ -algebra*, which is a  $C^*$ -algebra generated by the Toeplitz algebra  $\mathbf{A}$  and a single composition operator on  $H^2$  (See, for example, [10] and [11]) since

$$T_{\lambda,\varphi} = U_\lambda T_{\varphi_{\lambda,+}},$$

where  $T_{\varphi_{\overline{\lambda},+}}$  is the Toeplitz operator with symbol

$$\varphi_{\overline{\lambda},+} \sim \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \quad b_n = \overline{\lambda}^n a_n \text{ if } n \geq 0 \text{ and } b_n = a_n \text{ if } n < 0,$$

and the unitary

$$U_{\lambda} e_n = \lambda^n e_n, \quad n = 0, 1, 2, \dots$$

is clearly the same with the composition operator on  $H^2$  induced by  $\lambda z$  (For information on the theory of composition operators on Hardy spaces, we refer the readers to [4]). On the other hand, when  $|\lambda| < 1$ ,  $\lambda$ -Toeplitz operators are no longer Toeplitz-composition operators, but they can be written as the sum of weighted composition operators and their adjoints (See the discussion below following Proposition 2.2).

In this paper we study some basic properties, such as boundedness and compactness, for  $\lambda$ -Toeplitz operators and while doing so, we improve a result concerning the compactness for certain weighted composition operators (Proposition 2.3). Our main result, on the other hand, is to give necessary and sufficient conditions for the parameter  $\lambda$  and the symbol  $\varphi$  so that the corresponding  $\lambda$ -Toeplitz operator has finite rank (Proposition 3.1).

## 2 Bounded and compact $\lambda$ -Toeplitz operators

We shall proceed in two cases:  $|\lambda| = 1$  and  $|\lambda| < 1$ .

$|\lambda| = 1$ :

For  $\varphi \sim \sum a_n e^{in\theta}$  in  $L^2$ , let us recall the unitary  $U_{\lambda}$ , the function

$$\varphi_{\overline{\lambda},+} \sim \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \quad b_n = \overline{\lambda}^n a_n \text{ if } n \geq 0 \text{ and } b_n = a_n \text{ if } n < 0,$$

defined previously and the relation  $T_{\lambda,\varphi} = U_{\lambda} T_{\varphi_{\overline{\lambda},+}}$ . Hence by the classical results of Toeplitz operators, we have

**Proposition 2.1** *Let  $|\lambda| = 1$  and  $\varphi \sim \sum a_n e^{in\theta}$  in  $L^2$ .*

1. *The  $\lambda$ -Toeplitz operator  $T_{\lambda,\varphi}$  is bounded if and only if the Toeplitz operator  $T_{\varphi_{\overline{\lambda},+}}$  is bounded or, equivalently,  $\varphi_{\overline{\lambda},+} \in L^{\infty} = L^{\infty}(\mathbb{T})$ . Moreover, in this case, we have*

$$\|T_{\lambda,\varphi}\| = \|\varphi_{\overline{\lambda},+}\|_{\infty}.$$

2.  $T_{\lambda,\varphi}$  is compact if and only if  $\varphi_{\overline{\lambda},+} \equiv 0$ , i.e., if and only if  $\varphi \equiv 0$ .

**Remark.** We also have  $\|T_{\lambda,\varphi}\| = \|T_{\lambda,\varphi}U_{\overline{\lambda}}\| = \|T_{\varphi_{\overline{\lambda},-}}\|$  (hence  $\|T_{\lambda,\varphi}\| = \|\varphi_{\overline{\lambda},-}\|_{\infty}$ ), where  $T_{\varphi_{\overline{\lambda},-}}$  is the Toeplitz operator with symbol

$$\varphi_{\overline{\lambda},-} \sim \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \quad b_n = a_n \text{ if } n \geq 0 \text{ and } b_n = \overline{\lambda}^n a_n \text{ if } n < 0.$$

Note also that the boundedness of  $\varphi$  alone, not like the case in the Toeplitz operator  $T_{\varphi}$ , does not guarantee the boundedness of  $T_{\lambda,\varphi}$ . Here is an example: Consider the  $2\pi$ -periodic extension of  $\varphi(\theta) = \theta$ ,  $0 \leq \theta < 2\pi$  with Fourier series

$$1 + \sum_{n \neq 0} \frac{1}{in} e^{in\theta}.$$

Obviously  $\varphi \in L^{\infty}$ . But it is not difficult to see that  $T_{-1,\varphi}$  is not bounded since the function  $\varphi_{-1,+} \notin L^{\infty}$ . In fact in this case, we have  $T_{\lambda,\varphi}$  is bounded (or, equivalently,  $\varphi_{\overline{\lambda},+} \in L^{\infty}$ ) if and only if  $\lambda = 1$ .

$|\lambda| < 1$ :

The boundedness and compactness of  $T_{\lambda,\varphi}$  in this case are, basically, automatic:

**Proposition 2.2** *Let  $|\lambda| < 1$  and  $\varphi$  is measurable on  $\mathbb{T}$ . Then the  $\lambda$ -Toeplitz operator  $T_{\lambda,\varphi}$  is bounded if and only if  $\varphi \in L^2$ . In fact, every bounded  $\lambda$ -Toeplitz operator is Hilbert-Schmidt if  $|\lambda| < 1$ , and we have*

$$\|T_{\lambda,\varphi}\| \leq (1 - |\lambda|^2)^{-1/2} \left( \sum_{n=-\infty}^{\infty} |a_n|^2 \right)^{1/2},$$

where  $\varphi \sim \sum a_n e^{in\theta}$ .

**Proof** Just observe that when  $|\lambda| < 1$ ,

$$(1 - |\lambda|^2)^{-1} \left( \sum_{n=-\infty}^{\infty} |a_n|^2 \right) < \infty$$

if and only if  $\varphi \in L^2$ . ■

We will see later that  $T_{\lambda,\varphi}$  is actually in the trace class if  $|\lambda| < 1$ . The reason for the redundancy here is that we would like to compare what we have so far to a similar result from another class of operators on  $H^2$ . Let  $\tau$  be an analytic map from  $\mathbb{D}$  into  $\mathbb{D}$ . Then, given  $\psi$  analytic on  $\mathbb{D}$ , the weighted composition operator  $W_{\psi,\tau}$  on  $H^2$  is defined by

$$W_{\psi,\tau}f := \psi \cdot (f \circ \tau).$$

Weighted composition operators on Hardy spaces have been receiving increasing attention from operator theorists in the past decade on subjects including boundedness, compactness, spectrum (see, for example, [1], [2], [3], [6], [7], [12], [13]). In particular, on the subject of boundedness and compactness, results so far have suggested that  $W_{\psi,\tau}$  may be bounded, or even compact with unbounded  $\psi$ . For example, G. Gunatillake proves that  $W_{\psi,\tau}$  is bounded and in fact, compact, if  $\overline{\tau(\mathbb{D})} \subseteq \mathbb{D}$  ( $\overline{\tau(\mathbb{D})}$  being the closure of  $\tau(\mathbb{D})$ ) and  $\psi \in H^2$  by showing  $\|W_{\psi,\tau}|_{z^n H^2}\|$  tends to zero as  $n \rightarrow \infty$  (see Theorem 2, [7]). This of course implies immediately that  $T_{\lambda,\varphi}$  is compact if  $|\lambda| < 1$  since both  $W_{\varphi_+,\lambda z}$  and  $W_{\overline{\varphi}_-,\overline{\lambda}z}$  are compact and

$$T_{\lambda,\varphi} = W_{\varphi_+,\lambda z} + W_{\overline{\varphi}_-,\overline{\lambda}z}^*,$$

where  $\varphi_+ = P\varphi$  and  $\overline{\varphi}_-$  is the  $H^2$  function whose boundary value is given by the conjugate of  $\varphi_- = (I - P)\varphi$ , i.e.,

$$\overline{\varphi}_-(z) = \sum_{n=1}^{\infty} \overline{a}_{-n} z^n, \quad |z| < 1.$$

So it would appear that  $T_{\lambda,\varphi}$  being compact if  $|\lambda| < 1$  is just a special case and therefore a predictable result of Gunatillake's work, and its being Hilbert-Schmidt is merely a coincidence under special circumstances.

However, we will show that these "special circumstances" are not as special as one would think. In fact, we will show that the "Hilbert-Schmidt" conclusion also holds under Gunatillake's more general assumptions:

**Proposition 2.3** *Suppose that  $\overline{\tau(\mathbb{D})} \subseteq \mathbb{D}$ . Then  $W_{\psi,\tau}$  is Hilbert-Schmidt if  $\psi \in H^2$ .*

**Proof** The key is to express  $W_{\psi,\tau}$  in the form of an integral operator and analyze its kernel. Let us consider the reproducing kernel  $K_{\alpha}(z) = (1 - \overline{\alpha}z)^{-1}$  on  $H^2$ . Then, by the assumption that  $\overline{\tau(\mathbb{D})} \subseteq \mathbb{D}$ , the function

$$\psi(e^{i\phi}) \overline{K_{\tau(e^{i\phi})}(e^{i\theta})} = \psi(e^{i\phi})(1 - e^{-i\theta}\tau(e^{i\phi}))^{-1} \in L^2(\mathbb{T} \times \mathbb{T}).$$

Then since for  $f \in H^2$ ,

$$(W_{\psi,\tau}f)(z) = \psi(z)f(\tau(z)) = \psi(z)\langle f, K_{\tau(z)} \rangle, \quad |z| < 1,$$

it is not difficult to recognize that  $W_{\psi,\tau}$  is actually the same with the restriction of the  $L^2$  integral operator

$$f(e^{i\phi}) \longrightarrow \frac{1}{2\pi} \int_0^{2\pi} \psi(e^{i\phi}) \overline{K_{\tau(e^{i\phi})}(e^{i\theta})} f(e^{i\theta}) d\theta$$

on the boundary values of functions in  $H^2$ . This completes the proof since it is well-known that integral operators on  $L^2(\mathbb{T})$  with square integrable kernels are Hilbert-Schmidts.  $\blacksquare$

**Remark.** With the above conventions,  $T_{\lambda,\varphi}$  can be regarded as the integral operator

$$f(e^{i\phi}) \longrightarrow \frac{1}{2\pi} \int_0^{2\pi} (\varphi_+(e^{i\phi}) + \varphi_-(e^{i\theta})) \overline{K_{\lambda e^{i\phi}}(e^{i\theta})} f(e^{i\theta}) d\theta, \quad f \in H^2.$$

We now finish this section with

**Theorem 2.4**  $T_{\lambda,\varphi}$  belongs to the trace class if  $|\lambda| < 1$ .

**Proof** Let us consider the block form for the matrix of  $T_{\lambda,\varphi}$ :

$$\begin{pmatrix} A_m & B_m \\ C_m & D_m \end{pmatrix},$$

where  $A_m$  is the upper-left  $m \times m$  minor of the matrix. Since  $\|D_m\| = |\lambda|^m \|T_{\lambda,\varphi}\|$  and the rank of

$$\begin{pmatrix} A_m & B_m \\ C_m & 0 \end{pmatrix}$$

is at most  $2m$ , we see that the  $(2m+1)$ -th singular value of  $T_{\lambda,\varphi}$  is less than or equal to  $|\lambda|^m \|T_{\lambda,\varphi}\|$ . This completes the proof since the sequence of the singular values is decreasing.  $\blacksquare$

### 3 $\lambda$ -Toeplitz operators with finite rank

Basically the results in Section 2 tells us that if  $|\lambda| = 1$ , then  $T_{\lambda,\varphi}$  is never compact except in the trivial case and if, on the other hand,  $|\lambda| < 1$ , then any choice of  $\varphi$  in  $H^2$  will make  $T_{\lambda,\varphi}$  compact. Let us now consider a question in a some what similar nature: Given  $|\lambda| < 1$ , how can we tell whether  $T_{\lambda,\varphi}$  is of finite rank or not by the choice of  $\varphi$ ? We answer this question in the following:

**Proposition 3.1** *Let  $|\lambda| < 1$  and  $\varphi \sim \sum a_n e^{in\theta} \in L^2$ . Then  $T_{\lambda,\varphi}$  is of finite rank if and only if  $\varphi = 0$  or  $\lambda = 0$ .*

**Proof** Since  $T_{0,\varphi}$  has rank at most two, we will assume, from now on, that  $0 < |\lambda| < 1$ .

First we show that if  $\varphi_+ = 0$  or  $\varphi_- = 0$ , i.e., if the matrix of  $T_{\lambda,\varphi}$  is either lower triangular or upper triangular, then  $\dim \mathcal{R}(T_{\lambda,\varphi}) = \infty$  unless  $\varphi = 0$  (here  $\mathcal{R}(A)$  is the range of  $A$ ). Let us assume, without loss of generality,  $\varphi \neq 0$  and  $\varphi_- = 0$ . This means that  $\varphi_+ \neq 0$  and therefore  $T_{\lambda,\varphi}$  is the weighted composition operator  $W_{\varphi_+,\lambda z}$ , and hence  $S^{*n_0}T_{\lambda,\varphi}$  is the weighted composition operator  $W_{\psi,\lambda z}$ , where  $n_0 = \min\{n : a_n \neq 0\}$  and  $\psi = S^{*n_0}\varphi_+$ . Now suppose that  $\dim \mathcal{R}(T_{\lambda,\varphi}) < \infty$ . Then  $\dim \mathcal{R}(W_{\psi,\lambda z}) < \infty$ . This leads to a contradiction since the spectrum of  $W_{\psi,\lambda z}$ , according to Theorem 1 in [6], is the set

$$\{\lambda^k \psi(0) : k = 0, 1, 2, \dots\} \cup \{0\}$$

and this set is infinite since  $\psi(0) = a_{n_0} \neq 0$  and  $\lambda \neq 0$ .

What follows from the above discussion is that if  $a_n \neq 0$  for only finitely many nonnegative or negative  $n$ , then  $\dim \mathcal{R}(T_{\lambda,\varphi}) = \infty$ , unless  $a_n = 0$  for all  $n$  (i.e.  $\varphi = 0$ ). Indeed, for example, if  $a_n = 0$  for all but finitely many positive  $n$  and  $n_0 = \max\{n > 0 : a_n \neq 0\} \geq 0$ , then the matrix of the  $\lambda$ -Toeplitz operator  $S^{*n_0}T_{\lambda,\varphi}$  is upper triangular with nonzero diagonal, and hence  $\dim \mathcal{R}(T_{\lambda,\varphi}) = \infty$ . Consequently,  $T_{\lambda,\varphi}$  can not be of finite rank if at least one of  $\varphi_+$  and  $\varphi_-$  is a nonzero trigonometric polynomial.

Therefore, we shall proceed by assuming that  $\varphi_+$  and  $\varphi_-$  are nonzero and none of them are trigonometric polynomials. Since  $T_{\lambda,\varphi}$  has finite rank, the kernel of  $T_{\lambda,\varphi}$  contains nonzero polynomials (in  $H^2$ ). Hence by the identity

$$T_{\lambda,\varphi} = W_{\varphi_+,\lambda z} + W_{\bar{\varphi}_-,\bar{\lambda} z}^*,$$



we can find polynomials  $p, q \neq 0$  such that

$$\varphi_+(z)p(\lambda z) = q(\bar{\lambda}z), \quad |z| < 1 \quad (q = -P(\varphi_-p)).$$

Notice the above equality also depends on the fact that the composition operator  $C_{\bar{\lambda}z}$  and  $P$  commute. It follows that  $\varphi_+$  is a rational function. However, since  $\varphi_+$  is also in  $H^2$ , all poles of  $\varphi_+$  must lie strictly outside the unit disc (i.e., in the region  $\{z : |z| > 1\}$ ). So  $\varphi_+$  is actually continuous on the closed disc  $\bar{\mathbb{D}}$ . By considering  $T_{\lambda,\varphi}^*$ , we see that the same also holds for  $\bar{\varphi}_-$ .

Now, without loss of generality, we assume that  $\lambda > 0$  since otherwise, if  $\lambda = |\lambda|e^{i\alpha}$ , we have  $T_{\lambda,\varphi} = U_{e^{i\alpha}}T_{|\lambda|,\varphi_{e^{-i\alpha},+}}$ , and note that

$$\varphi_+(z) = (\varphi_{e^{-i\alpha},+})_+(e^{i\alpha}z).$$

Let  $\tilde{\varphi}$  be the function on  $\mathbb{T}$  defined by

$$\tilde{\varphi}(e^{i\theta}) = \varphi_+(e^{i\theta}) + \overline{\varphi_-(\lambda e^{i\theta})}, \quad e^{i\theta} \in \mathbb{T}.$$

So  $\tilde{\varphi}$  is continuous on  $\mathbb{T}$ , and hence  $T_{\tilde{\varphi}}$  is a bounded Toeplitz operator. On the other hand, again by the fact that  $C_{\lambda z}$  commute with  $P$ , it is not difficult to check that (remembering  $\lambda > 0$ )

$$T_{\lambda,\varphi} = T_{\tilde{\varphi}}C_{\lambda z}.$$

But because  $C_{\lambda z}$  is one to one, the fact that  $T_{\lambda,\varphi}$  has finite rank implies that  $T_{\tilde{\varphi}}$  also has finite rank, hence  $\tilde{\varphi} = 0$  and, consequently,  $\varphi = 0$ . ■

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